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On wave-like differential equations in general Hilbert space with application to Euler–Bernoulli bending vibrations of a beam

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ABSTRACT

Wave-like differential equations occur in many engineering applications. Here the engineering setup is embedded into the framework of functional analysis of modern mathematical physics. After an overview, the L^2–Hilbert space approach to free Euler–Bernoulli bending vibrations of a beam in one spatial dimension is investigated. We analyze in detail the corresponding positive, selfadjoint differential operators of 4-th order associated to the boundary conditions in statics. A comparison with free string wave swinging is outlined.

1. Introduction, overview

Wave-like differential equations occur in many engineering applications. In engineering textbooks the solution methods seem to be very specific to the specially chosen situation, leaving some deeper mathematical questions unanswered. Nevertheless the methods used are successful and appropriate for the selected application, leading to very concrete solutions, analytically and/or for numerical solving procedures, e.g. Refs. 1–4. In case of beam dynamics four engineering theories exist: Euler–Bernoulli model, Rayleigh model, shear model and Timoshenk model. Classically the dynamics of the transversally bending beam is investigated by eigenfunction expansion.[5]

Functional analysis is capable to provide general statements for very general cases, namely predictions on existence and smoothness degrees of eigenfunctions and solutions (regularity). Furthermore even numerical solutions of partial differential equations can be obtained with methods from functional analysis as the reproducing kernel Hilbert space method, see e.g. Refs. 6–8. The mentioned specific engineering techniques are far from being able to deal with this generality. The present article aims to bring together computational and theoretical engineering science with functional analysis as used in modern mathematical physics. The novelty of our approach is the complete incorporation and investigation of free Euler–Bernoulli vibrations in the general context of Hilbert space operator theory in functional analysis.

In order to be precise let us introduce now, what will be understood under a wave-like differential equation in Hilbert space language.

Definition 1.1 (*Wave-Like Differential Equation*). A differential equation of type $\frac{d^2u(t)}{dt^2} = -Au(t)$ in a Hilbert space $\mathcal{H}$ with some positive, selfadjoint operator A is called to be *wave-like*. A solution of which is a trajectory $\mathbb{R} \ni t \mapsto u(t) \in \mathcal{H}$, where the variable $t \in \mathbb{R}$ is interpreted as evolution in time.

In engineering or physical applications, the operators A usually represent differential operators of second or higher order acting in some L^2–Hilbert space of square integrable functions on a region $\Lambda \subseteq \mathbb{R}^r$, $r \in \{1,2,3\}$. The primary wave equation concerns Laplace operators $A = -\Delta$, so the notion *wave-like* is a generalization. To free Euler–Bernoulli bending vibrations of a beam belong differential operators A of 4th order.

We treat first the general Hilbert space solution of wave-like differential equations, Section 2. In Section 3 the mathematical procedure for L^2–Hilbert space approach is described and some general results

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for related differential operators from the literature are reported. Moreover, a short overview over three wave-like equations is given: primary wave dynamics, wave decoupling for electromagnetic radiation, and free bending vibrations of a plate.

In Section 4 we outline in detail the L^2–Hilbert space frame for positive, selfadjoint differential operators A of 4th order necessary for describing the free Euler–Bernoulli bending vibrations of a beam in the interval $(0,\ell)$ of length $\ell > 0$ in terms of wave-like equations. Intrinsically involved into the domain of definition of such a differential operator A of 4th order is the chosen boundary condition (support) of the beam. Each A is identified as the well known Friedrichs extension[9,10] of the suitable product operator of four differential operators of first order respecting exactly the support of the beam. For such positive, selfadjoint A we prove the existence of a purely discrete spectrum with help of a Sobolev compact embedding theorem. Two groups of these operators A are distinguished, one group with analytically solvable eigenequations, and the other group for which only numerical solutions are possible; that reflects directly some properties of the mentioned Friedrichs extensions and presents unknown operator features.

In Section 5 we discuss the difference between free bending vibrations and free string wave swinging for comparable boundary conditions.

The detailed mathematical proofs are given in the last Section 6.

Abbreviating we write IV for initial value(s), IVP for initial value problem, BV for boundary value(s), IBVP for initial boundary value problem, PDE for partial differential equation(s), and ONB for orthonormal basis.

Moreover, the natural numbers are without zero, namely $\mathbb{N} = \{1,2,3,4,\dots\}$.

2. Wave-like differential equation in general Hilbert space

Let $\mathcal{H}$ be a separable real or complex Hilbert space with inner (scalar) product $\langle .|.\rangle$, being anti-linear in the first and linear in the second variable in the complex case, and with associated norm $\|.\| = \sqrt{\langle .|.\rangle}$. (Note: In some purely mathematical texts the inner product is taken linear in the first factor, but linearity in the second factor is general standard in mathematical physics. Also the notion $\langle .|.\rangle$ for the scalar product is common in mathematical physics, but in mathematics one also finds $(.,.)$ or $(.;.)$.)

All operators used are linear, so we will not mention this anymore. Discontinuity of an operator A (in the $\|.\|$–topology on $\mathcal{H}$) is equivalent to its unboundedness. For such an unbounded operator A its domain of definition $\mathrm{dom}(A)$ cannot be the whole Hilbert space $\mathcal{H}$.

For an operator A with (norm-) dense domain of definition $\mathrm{dom}(A) \subseteq \mathcal{H}$, there are the following notions in Hilbert space theory:

(a) A is called *positive*, if $\langle \xi|A\xi\rangle \geq 0$ for all $\xi \in \mathrm{dom}(A)$, denoted as $A \geq 0$.

(b) The *adjoint* A^* of A is defined by (η_ξ is unique since $\mathrm{dom}(A)$ is dense)

$$\mathrm{dom}(A^*) = \{\xi \in \mathcal{H} \mid \exists\, \eta_\xi \in \mathcal{H} \text{ with } \langle \eta_\xi|\varphi\rangle = \langle \xi|A\varphi\rangle\ \forall \varphi \in \mathrm{dom}(A)\},$$
$$A^*\xi = \eta_\xi\,, \qquad \forall \xi \in \mathrm{dom}(A^*)\,. \tag{2.1}$$

(c) A is *symmetric*, if $A\xi = A^*\xi$ for all $\xi \in \mathrm{dom}(A) \subseteq \mathrm{dom}(A^*)$, denoted by $A \subseteq A^*$, that is, if $\langle \xi|A\varphi\rangle = \langle A\xi|\varphi\rangle$ for all $\xi, \varphi \in \mathrm{dom}(A)$.

(d) A is called *selfadjoint*, if $A = A^*$, i.e. symmetry with $\mathrm{dom}(A) = \mathrm{dom}(A^*)$.

Selfadjointness (not only symmetry) of an operator A is an important property, since only such operators enable spectral calculus, e.g. Refs. 10, 11: Each real- or complex-valued function $f : \sigma(A) \ni y \mapsto f(y) \in \mathbb{R}$ or $\mathbb{C}$, being defined on the spectrum $\sigma(A) \subseteq \mathbb{R}$ of A, gives rise to an operator $f(A)$ acting on $\mathcal{H}$. $f(A)$ is selfadjoint, if and only if the ordinary function f is real-valued. $f(A)$ is a bounded, thus continuous operator, if f is a bounded function. $f(A)$ is a positive operator on $\mathcal{H}$, if f has values only in the positives $[0,\infty)$. Note that A is positive if and only if its spectrum is positive, that is $\sigma(A) \subseteq [0,\infty)$.

Theorem 2.1 (*Wave-Like IVP*). *Consider the following IVP for the positive, selfadjoint operator A in the Hilbert space $\mathcal{H}$ and given vectors $u_0, \dot{u}_0 \in \mathcal{H}$,*

$$\text{differential equation} \qquad \frac{d^2u(t)}{dt^2} = -Au(t)\,, \quad t \in \mathbb{R}\,, \tag{2.2}$$
$$\text{IV (at } t=0) \qquad u(t)|_{t=0} = u_0 \in \mathcal{H}\,, \quad \left.\frac{du(t)}{dt}\right|_{t=0} = \dot{u}_0 \in \mathcal{H}\,,$$

(The differential equation (2.2) is short notion, it is mathematically rigorously formulated in the weak sense as $\frac{d^2}{dt^2}\langle \eta|u(t)\rangle = -\langle A\eta|u(t)\rangle$ for all $\eta \in \mathrm{dom}(A)$.)

Then the unique solution trajectory of the wave-like IVP is given by

$$u(t) = \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}\,\dot{u}_0\,, \quad \forall t \in \mathbb{R}\,. \tag{2.3}$$

Moreover, the trajectory $\mathbb{R} \ni t \mapsto u(t) \in \mathcal{H}$ is continuous with respect to the norm $\|.\|$ on $\mathcal{H}$.

Sketch of Proof. That (2.3) is indeed a solution of the IVP, is immediately verified with help of the spectral calculus. For uniqueness see Refs. 12, 13. ■

Remark, for each $t \in \mathbb{R}$ the selfadjoint operators $\cos(t\sqrt{A})$ and $\frac{\sin(t\sqrt{A})}{\sqrt{A}}$ arise by spectral calculus from the ordinary continuous bounded functions of a single variable

$$[0,\infty) \ni y \mapsto \cos(t\sqrt{y})\,, \qquad [0,\infty) \ni y \mapsto \begin{cases} t\,, & \text{if } y = 0\,, \\ \frac{\sin(t\sqrt{y})}{\sqrt{y}}\,, & \text{if } y > 0\,. \end{cases} \tag{2.4}$$

Regardless of whether A is bounded or unbounded, both selfadjoint operators $\cos(t\sqrt{A})$ and $\frac{\sin(t\sqrt{A})}{\sqrt{A}}$ are bounded, thus defined everywhere in $\mathcal{H}$.

Corollary 2.2. *If $\mathcal{H}$ is a complex Hilbert space, then the solution trajectory $t \mapsto u(t)$ from (2.3) is related to the strongly continuous unitary group $e^{it\sqrt{A}}$ in the following sense: $u(t) = e^{it\sqrt{A}}u_0$ for all $t \in \mathbb{R}$, if and only if $\dot{u}_0 = i\sqrt{A}\,u_0$.*

In the next sections we consider such positive, selfadjoint operators A for physical or technical applications. There A is often modified to $\varsigma^2 A$ with some physical or material constant $\varsigma > 0$. Then $\sqrt{A}$ has to be replaced by $\varsigma\sqrt{A}$.

Let us suppose that the positive, selfadjoint operator A possesses a pure point (= purely discrete) spectrum, $\sigma(A) = \sigma_p(A) \subset [0,\infty)$. Then there exists an ONB of $\mathcal{H}$ consisting of normalized eigenvectors ψ_n, $n \in \mathbb{N}$ (since $\mathcal{H}$ is supposed to be separable, the ONB is countable), corresponding to the eigenvalues (= discrete spectral points) $a_n \geq 0$, $n \in \mathbb{N}$, that is

$$A\psi_n = a_n\psi_n \qquad \Rightarrow \qquad f(A)\psi_n = f(a_n)\psi_n\,, \qquad \forall n \in \mathbb{N}\,. \tag{2.5}$$

Corollary 2.3. *With the purely discrete spectrum (2.5) of the positive, selfadjoint operator A, the solution $u(t)$ of formula (2.3) rewrites as*

$$u(t) = \sum_{n=1}^{\infty} \underbrace{\Big(\cos(t\sqrt{a_n})\langle \psi_n|u_0\rangle + \frac{\sin(t\sqrt{a_n})}{\sqrt{a_n}}\langle \psi_n|\dot{u}_0\rangle\Big)}_{=\ \langle \psi_n|u(t)\rangle}\psi_n\,, \quad \forall t \in \mathbb{R}\,. \tag{2.6}$$

Proof. Remark the spectral properties according to the second part of Eq. (2.5),

$$\cos(t\sqrt{A})\psi_n = \cos(t\sqrt{a_n})\psi_n\,, \qquad \frac{\sin(t\sqrt{A})}{\sqrt{A}}\psi_n = \frac{\sin(t\sqrt{a_n})}{\sqrt{a_n}}\psi_n\,. \tag{2.7}$$

Since the normalized eigenvectors ψ_n, $n \in \mathbb{N}$ constitute an ONB of $\mathcal{H}$, we may decompose $u(t)$ of (2.3) according to the spectral projections

$$\begin{aligned} u(t) &= \sum_{n=1}^{\infty} \langle \psi_n | u(t) \rangle \psi_n \\ &= \sum_{n=1}^{\infty} \langle \psi_n | \cos(t\sqrt{A}) u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}} \dot{u}_0 \rangle \psi_n \\ &= \sum_{n=1}^{\infty} \Big(\langle \psi_n | \cos(t\sqrt{A}) u_0 \rangle + \langle \psi_n | \frac{\sin(t\sqrt{A})}{\sqrt{A}} \dot{u}_0 \rangle \Big) \psi_n \\ &\stackrel{\star}{=} \sum_{n=1}^{\infty} \Big(\langle \cos(t\sqrt{A}) \psi_n | u_0 \rangle + \langle \frac{\sin(t\sqrt{A})}{\sqrt{A}} \psi_n | \dot{u}_0 \rangle \Big) \psi_n \\ &\stackrel{(2.7)}{=} \sum_{n=1}^{\infty} \Big(\cos(t\sqrt{a_n}) \langle \psi_n | u_0 \rangle + \frac{\sin(t\sqrt{a_n})}{\sqrt{a_n}} \langle \psi_n | \dot{u}_0 \rangle \Big) \psi_n , \end{aligned}$$

where at the equality sign $\stackrel{\star}{=}$ with star we used the selfadjointness of the bounded operators $\cos(t\sqrt{A})$ and $\frac{\sin(t\sqrt{A})}{\sqrt{A}}$. ■

3. On applications in L^2–Hilbert spaces

Let $\mathcal{H} = \mathrm{L}^2(\Lambda)$ be the Hilbert space of $\mathbb{R}$– or $\mathbb{C}$–valued, Lebesgue square integrable functions defined on the subset $\Lambda \subseteq \mathbb{R}^r$ in $r \in \mathbb{N}$ real dimensions, with standard inner product ($\overline{\xi(x)}$ complex conjugate to $\xi(x)$) and norm,

$$\langle \xi | \eta \rangle = \int_\Lambda \overline{\xi(x)} \eta(x)\, d^r x\,, \quad \|\xi\|^2 = \langle \xi | \xi \rangle = \int_\Lambda |\xi(x)|^2 d^r x\,, \quad \forall \xi, \eta \in \mathrm{L}^2(\Lambda).$$

Λ is chosen as an *open* and *connected* subset of $\mathbb{R}^r$, which usually is called a *domain* or a *region*. “Connected” means “path connected”, so that any pair of points in Λ may be connected via a continuous path within Λ. The domain Λ is called *interior* if Λ is bounded, and *exterior* if its set complement $\mathbb{R}^r \setminus \Lambda$ is bounded. $\bar{\Lambda}$ denotes the topological closure of Λ, and, $\partial\Lambda = \bar{\Lambda} \setminus \Lambda$ its boundary.

The first step is to transform the spatial differentiation operation on Λ into a positive, selfadjoint operator A acting on the Hilbert space $\mathrm{L}^2(\Lambda)$, a procedure, which in general requires much mathematical effort. That operator A is taken for the wave-like differential equation (2.2). The exact mathematical definition of such a selfadjoint differential operator A on $\mathrm{L}^2(\Lambda)$ is often done in terms of a positive sesquilinear form, which intrinsically includes the considered BV, for examples see e.g. Refs. 9, 11–15 and Section 4. Sometimes the considered BV requires some kind of smoothness for the boundary $\partial\Lambda$ of the region Λ, e.g. segment property, or uniform cone property, or piece-wise C^k–smoothness, etc.

3.1. Primary wave PDE in a spatial region Λ

Consider the Laplacian on an arbitrary domain $\Lambda \subseteq \mathbb{R}^r$,

$$-\Delta = -\left(\partial_1^2 + \partial_2^2 + \cdots + \partial_r^2 \right),$$

where $\partial_j = \frac{\partial}{\partial x_j}$ for $j = 1, 2, \ldots, r$. It is well known that the Laplacian $-\Delta$ indeed gives rise to a positive, selfadjoint operator A acting on $\mathrm{L}^2(\Lambda)$ for each of the classical homogeneous BV such as Dirichlet, or Neumann, or mixed. Even when incorporating an anisotropic, inhomogeneous medium into Λ, in many cases positivity and selfadjointness hold, e.g. Refs. 12–14.

Here (2.2) describes the classical propagating wave in Λ satisfying the BV for which the positive, selfadjoint Laplacian $A = -\Delta$ is defined. In terms of continuously differentiable functions $u(t)(x_1, x_2, \ldots, x_r) = u(x_1, x_2, \ldots, x_r, t)$, the differential equation (2.2) is rewritten as the well known wave PDE with wave velocity $\varsigma > 0$, namely

$$\partial_t^2 u(x_1, \ldots, x_r, t) = -\varsigma^2 (-\Delta) u(x_1, \ldots, x_r, t).$$

3.2. Maxwell radiation and wave equations in electromagnetism

Details to the present subsection are found in Ref. 15. We consider electromagnetism in vacuum in the spatial region $\Lambda \subseteq \mathbb{R}^3$. We use the real Hilbert space $\mathrm{L}^2(\Lambda, \mathbb{R}^6)$ for $\mathbb{R}^6$–valued functions on Λ (vector functions with 6 components). $\mathbf{E}(t)$ denotes the electric field and $\mathbf{B}(t)$ the magnetic field (3 components for each), depending on time t. Assuming no current and no charge distribution in Λ the two dynamical Maxwell equations are summarized in matrix notation as

$$\frac{d}{dt} \underbrace{\begin{pmatrix} \mathbf{E}(t) \\ \mathbf{B}(t) \end{pmatrix}}_{= u(t)} = \underbrace{\begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl}_0 & 0 \end{pmatrix}}_{= \mathbb{A}} \underbrace{\begin{pmatrix} \mathbf{E}(t) \\ \mathbf{B}(t) \end{pmatrix}}_{= u(t)} . \tag{3.1}$$

The dielectric constant ϵ_0 and the magnetic permeability μ_0 are set to 1 for convenience. The walls of Λ, i.e. the boundary $\partial\Lambda$, are supposed to consist of a perfect conductor material. This leads to the well known BV $\mathbf{E}(t) \times n|_{\partial\Lambda} = 0$ and $\mathbf{B}(t) \cdot n|_{\partial\Lambda} = 0$, where n denotes the outer normal vector at the boundary points. The two curl (rotation) operators, curl_0 and curl, are well defined according to these BV by minimal and maximal Sobolev domains of definition, respectively.

The Maxwell operator $\mathbb{A}$ is anti-selfadjoint, meaning $\mathbb{A}^* = -\mathbb{A}$ (in the complexified Hilbert space $i\mathbb{A}$ is selfadjoint), since $\mathrm{curl}^* = \mathrm{curl}_0$ and $\mathrm{curl}_0^* = \mathrm{curl}$ for the adjoints. Consequently, $\exp\{t\mathbb{A}\}$, $t \in \mathbb{R}$, constitutes a strongly continuous orthogonal group in $\mathrm{L}^2(\Lambda, \mathbb{R}^6)$. With IV $u(t)|_{t=0} = u_0 = \begin{pmatrix} \mathbf{E}_0 \\ \mathbf{B}_0 \end{pmatrix}$ the unique solution trajectory of the IVP (3.1) is given by

$$u(t) = \exp\{t\mathbb{A}\} u_0\,, \quad \forall t \in \mathbb{R}\,. \tag{3.2}$$

(3.2) describes the freely evolving electromagnetic field in the spatial region Λ, namely the radiation, in which intense coupling of the electric and magnetic fields takes place due to the non-diagonal matrix operator $\mathbb{A}$ in (3.1).

It is well known that the electric and the magnetic components can be decoupled. Second time derivative in (3.1) leads to the wave-like equation

$$\frac{d^2 u(t)}{dt^2} = \mathbb{A}^2 u(t) = -\underbrace{\mathbb{A}^* \mathbb{A}}_{\geq 0} u(t)\,, \tag{3.3}$$

where we have inserted $\mathbb{A}^* = -\mathbb{A}$. It holds that

$$\begin{aligned} \mathbb{A}^*\mathbb{A} = -\mathbb{A}^2 &= -\begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl}_0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl}_0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{curl}\,\mathrm{curl}_0 & 0 \\ 0 & \mathrm{curl}_0\,\mathrm{curl} \end{pmatrix} \end{aligned}$$

is a diagonal matrix operator in the electromagnetic field Hilbert space $\mathrm{L}^2(\Lambda, \mathbb{R}^6)$, which decouples the electric and magnetic fields. $\mathbb{A}^*\mathbb{A}$ is a positive, selfadjoint operator, and so are both double curl operators $\mathrm{curl}\,\mathrm{curl}_0$ and $\mathrm{curl}_0\,\mathrm{curl}$.

Decoupling ensures that we get two separate wave equations, one for the electric field and another for the magnetic field, each of which lives now in the Hilbert space $\mathrm{L}^2(\Lambda, \mathbb{R}^3)$ with three components, only,

$$\frac{d^2\mathbf{E}(t)}{dt^2} = -\overbrace{\mathrm{curl}\,\mathrm{curl}_0}^{-\Delta_E \geq 0} \mathbf{E}(t)\,, \qquad \frac{d^2\mathbf{B}(t)}{dt^2} = -\overbrace{\mathrm{curl}_0\,\mathrm{curl}}^{-\Delta_B \geq 0} \mathbf{B}(t)\,. \tag{3.4}$$

Here the operators $\mathrm{curl}\,\mathrm{curl}_0$ and $\mathrm{curl}_0\,\mathrm{curl}$ agree with two different Laplace operators $-\Delta_E$ and $-\Delta_B$ not covered by the mentioned classical BV cases in the previous subsection. Solutions of both wave Eqs. (3.4), or equivalently of (3.3), agree with the original solution (3.2) of the dynamic Maxwell equations (3.1) only for the correlation of the IV in direct analogy to Corollary 2.2,

$$\dot{u}_0 = \begin{pmatrix} \dot{\mathbf{E}}_0 \\ \dot{\mathbf{B}}_0 \end{pmatrix} = \left.\frac{du(t)}{dt}\right|_{t=0} = \mathbb{A} u_0 = \begin{pmatrix} \mathrm{curl}\,\mathbf{B}_0 \\ -\mathrm{curl}_0\,\mathbf{E}_0 \end{pmatrix} . \tag{3.5}$$

3.3. Free bending vibrations of a plate

A plate is in $\Lambda \subseteq \mathbb{R}^r$, and bends into a further (spatial) dimension. For an isotropic, homogeneous plate the differential operator A is given up to some material constant by $-\Delta^2 = -\big(\partial_1^2 + \cdots + \partial_r^2\big)^2$. In the literature one finds some BV, which give rise to positivity and selfadjointness of A, see e.g. Ref. 12.

The literature however does not cover the case $r = 1$ with the different BV from statics, which we will investigate in great mathematical detail in Section 4.

3.4. On spectral properties of the differential operators A

Let Λ be interior, with some mild assumptions about the smoothness of the boundary $\partial\Lambda$ when necessary. Then a positive, selfadjoint differential operator A of the above mentioned types has a pure point (= purely discrete) spectrum $\sigma(A) = \sigma_p(A) \subset [0,\infty)$. Each eigenspace is finite dimensional and the eigenvalues a_n from (2.5) may be arranged increasingly and converge to infinity,

$$0 \le a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le \cdots .. , \qquad \lim_{n\to\infty} a_n = \infty . \tag{3.6}$$

Exceptions are the two curlcurl–operators in Section 3.2, since the kernels of these operators are infinite dimensional. But for the strictly positive eigenvalues the preceding statements (3.6) remain valid. The existence of such a discrete spectrum is proven with help of compact embeddings of related Sobolev spaces into $L^2(\Lambda)$, e.g. Refs. 13, 14. We will do so in Section 6, too. However, analytic calculations of eigenfunctions are only possible, if Λ has certain geometric properties, e.g. such as parallelepiped, ball, circular disc, or bounded interval.

When Λ is exterior, then A possesses an absolutely continuous spectrum $\sigma(A) = \sigma_{ac}(A) = [0,\infty)$, which is necessary for scattering theory, e.g. Ref. 11, etc.

3.5. On regularity of eigenvectors and solution trajectories

Every Hilbert space element $v \in L^2(\Lambda)$ represents a class of functions defined on Λ, which agree *almost everywhere* in Λ. Thus point evaluations and notions of continuity or partial differentiation do not make sense. Therefore an L^2–Hilbert space formulation of an ordinary PDE is a generalization.

A function $u : \Lambda \to \mathbb{R}$ or $\mathbb{C}$ is called smooth, if it is (partially) continuous or continuously differentiable in some degree. When a Hilbert space class $v \in L^2(\Lambda)$ contains a function u with some smoothness properties, then the class v usually is represented by that smooth representant u.

Regularity statements, e.g. Refs. 12–14, are of the following kind: Eigenvectors $\psi_n = \psi_n(x_1,\dots,x_r)$ of a differential operator A, and L^2–solution trajectories $u(t)$ in Theorem 2.1 possess some kind of smoothness, whenever the boundary $\partial\Lambda$ and the IV functions $u_0,\ \dot u_0 \in L^2(\Lambda)$ fulfill some degree of smoothness. Then in addition $u(t)$ satisfies the underlying ordinary PDE.

Each domain $\Lambda \subseteq \mathbb{R}$ in one dimension is a bounded or unbounded open interval and thus possesses a completely smooth boundary, so regularity arises.

4. Bending vibrations of a beam in one spatial dimension

In the interior open interval $\Lambda = (0,\ell)$ with boundary points $x = 0$ and $x = \ell$ is placed a slender, isotropic, homogeneous, straight, elastic beam of length ℓ with constant cross–sectional area. The x–axis is along the neutral fiber of the beam, and the bending deformations $u(t)(x) = u(x,t)$ are vertical (transversal) to the x–axis. It is assumed that the beam is supported only at its ends, namely at $x = 0$ and at $x = \ell$.

4.1. Sobolev spaces for the open interval $(0,\ell)$

For the mathematical description of differential operators it is inevitable to work with Sobolev spaces. We state here some properties needed subsequently, e.g. Ref. 13, overview in Ref. 15.

Let $L^2 = L^2((0,\ell))$ denote the complex Hilbert space of square integrable, $\mathbb{C}$–valued functions on $(0,\ell)$ with inner product and norm

$$\langle \xi|\eta\rangle = \int_0^\ell \overline{\xi(x)}\eta(x)\,dx\,, \quad \|\xi\|^2 = \langle \xi|\xi\rangle = \int_0^\ell |\xi(x)|^2 dx\,, \quad \forall \xi,\eta \in L^2.$$

By $C_c^\infty(I)$ we denote the set of infinitely often continuously differentiable functions $\xi : I \to \mathbb{C}$ for the open interval $I \subseteq \mathbb{R}$ with compact support within I, the standard test function space in distribution theory for the interval I. The elements of $C_c^\infty(I)|_J$ are the restrictions $\xi|_J$ of $\xi \in C_c^\infty(I)$ to the open subinterval $J \subseteq I$.

Let $\xi : \mathbb{R} \to \mathbb{C}$ be an s–times continuously differentiable function and φ a test function on $(0,\ell)$, that is $\varphi \in C_c^\infty((0,\ell))$. When integrating s–times partially, no boundary terms occur, since the test function φ has compact support in $(0,\ell)$ and hence vanishing boundary values $\varphi^{(k)}(0) = 0 = \varphi^{(k)}(\ell)$ for all derivatives,

$$\langle \xi^{(s)}|\varphi\rangle = \int_0^\ell \overline{\xi^{(s)}(x)}\varphi(x)\,dx = (-1)^s \int_0^\ell \overline{\xi(x)}\varphi^{(s)}(x)\,dx = (-1)^s\langle \xi|\varphi^{(s)}\rangle.$$

This is the guiding line for the definition of *square integrable distributional differentiability*: Suppose for a $\xi \in L^2$ the existence of a vector $\xi^{(s)} \in L^2$ with

$$\langle \xi^{(s)}|\varphi\rangle = (-1)^s\langle \xi|\varphi^{(s)}\rangle\,, \quad \forall \varphi \in C_c^\infty((0,\ell))\,. \tag{4.1}$$

Then $\xi^{(s)} \in L^2$ is called the square integrable sth distributional derivative of $\xi \in L^2$. Provided existence, $\xi^{(s)}$ is unique, since $C_c^\infty((0,\ell))$ is dense in L^2. So, the distributional definition is an extension of ordinary differentiation using conventional differential limits.

Definition 4.1 (*Sobolev Spaces*). For each $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0,1,2,3,\dots\}$ the mth Sobolev space is defined as $W^m = W^m((0,\ell)) := \{\xi \in L^2 \mid \xi^{(s)} \in L^2 \text{ for } 0 \le s \le m\}$. It is equipped with the Sobolev inner product and norm

$$\langle \xi|\eta\rangle_m := \sum_{s=0}^m \langle \xi^{(s)}|\eta^{(s)}\rangle\,, \qquad \|\xi\|_m = \sqrt{\langle \xi|\xi\rangle_m}\,. \tag{4.2}$$

The index $m = 0$ yields $\langle \xi|\eta\rangle_0 = \langle \xi|\eta\rangle$ and $\|\xi\|_0 = \|\xi\| = \sqrt{\langle \xi|\xi\rangle}$, the conventional scalar product and norm on $L^2 = W^0$.

Proposition 4.2. *The following assertions are valid:*

- *(a) W^m is a separable complex Hilbert space for every $m \in \mathbb{N}_0$ with respect to its Sobolev inner product $\langle .|.\rangle_m$ and mth norm $\|.\|_m$, cf. Eq. (4.2).*
- *(b) $C_c^\infty(\mathbb{R})|_{(0,\ell)}$ is $\|.\|_m$–dense in the mth Sobolev space W^m for each $m \in \mathbb{N}_0$. (This way W^m may defined without distributional derivatives.)*
- *(c) If $m > k$, then $W^m \subseteq C^k([0,\ell])$, the k–times continuously differentiable functions on the open interval $(0,\ell)$, for which each derivative extends continuously to both boundary points $x = 0$ and $x = \ell$.*
- *(d) Let $m > n$. Then the identical embedding $W^m \hookrightarrow W^n$ is continuous, injective and a compact map. Especially, $W^1 \hookrightarrow L^2$ is compact.*

4.2. Differential operators of first order for different BV

We define four different differential operators $\delta_{..} \in \{\delta_{++},\delta_{+-},\delta_{-+},\delta_{--}\}$ acting on L^2 as

$$\delta_{..}\,\xi = \xi'\,, \quad \xi \in \operatorname{dom}(\delta_{..}) \subset L^2.$$

Unboundedness (discontinuity) makes it impossible that such a differential operator may act on all Hilbert space vectors. Therefore, we define four different $\|.\|$–dense domains of definition leading to four different operators. By Proposition 4.2(c) we know that W^1 is a subspace of $C^0([0,\ell])$ (continuous functions on $[0,\ell]$) and thus allows for a direct boundary evaluation,

$$\begin{aligned}
&\delta_{++} \quad \mathrm{dom}(\delta_{++}) := \{\xi \in W^1 \mid \xi(0)=0,\ \xi(\ell)=0\};\\
&\delta_{+-} \quad \mathrm{dom}(\delta_{+-}) := \{\xi \in W^1 \mid \xi(0)=0,\ \text{no BV at } \ell\};\\
&\delta_{-+} \quad \mathrm{dom}(\delta_{-+}) := \{\xi \in W^1 \mid \text{no BV at } 0,\ \xi(\ell)=0\};\\
&\delta_{--} \quad \mathrm{dom}(\delta_{--}) := \{\xi \in W^1 \mid \text{no BV at both } 0,\ell\} = W^1.
\end{aligned}$$

These domains are $\|.\|_1$–closed subspaces of W^1, since the embedding $W^1 \subseteq C^0([0,\ell])$ is continuous. The minus or plus sign in the index means for "+" that the BV zero is fulfilled, and for "−" the BV is not fulfilled, corresponding to the left or right boundary point, $x = 0$ and $x = \ell$, respectively.

Let us mention that these four differential operators of first order are auxiliary but necessary for introducing the correct BV from statics for the diverse differential operators of 4th order for the bending beam in Section 4.4.

Lemma 4.3. *The four differential operators $\delta_{..} \in \{\delta_{++},\delta_{+-},\delta_{-+},\delta_{--}\}$ are closed, and for their L^2–adjoints it holds that*

$$\delta^*_{++} = -\delta_{--}, \quad \delta^*_{--} = -\delta_{++}, \qquad \delta^*_{+-} = -\delta_{-+}, \quad \delta^*_{-+} = -\delta_{+-}.$$

Proof. The graph norm of these operators agrees with the Sobolev norm $\|.\|_1$. So they are closed unbounded operators in L^2 by construction. Especially

$$C_c^\infty((0,\ell)), \quad C_c^\infty((0,\infty))|_{(0,\ell)}, \quad C_c^\infty((-\infty,\ell))|_{(0,\ell)}, \quad C_c^\infty(\mathbb{R})|_{(0,\ell)}$$

are operator cores, which are $\|.\|_1$–dense in the domains $\mathrm{dom}(\delta_{..}) \subseteq W^1$, respectively.

Let us first consider the pair δ_{++} and δ_{--}. According to the construction of the adjoint of an operator in (2.1) we have

$$\mathrm{dom}(\delta^*_{++}) = \{\xi \in L^2 \mid \exists\, \eta_\xi \in L^2 \text{ with } \langle \eta_\xi|\varphi\rangle = \langle \xi|\varphi'\rangle\ \forall \varphi \in C_c^\infty((0,\ell))\},$$

and $\delta^*_{++}\xi := \eta_\xi$. It is sufficient to use here the core $C_c^\infty((0,\ell))$ of δ_{++}, instead of its whole domain of definition as in (2.1). That coincides with the concept of square integrable distributional differentiation from the previous subsection, see (4.1). Thus $\xi \in W^1 = \mathrm{dom}(\delta_{--})$, and moreover, $\mathrm{dom}(\delta^*_{++}) = W^1$ with $-\xi' = -\delta_{--}\xi = \delta^*_{++}\xi$, which means $\delta^*_{++} = -\delta_{--}$. The closedness of δ_{++} gives by adjoining, $\delta_{++} = \delta^{**}_{++} = -\delta^*_{--}$.

For δ_{+-} and δ_{-+} the situation is different. By definition of the adjoint it holds

$$\mathrm{dom}(\delta^*_{-+}) = \{\xi \in L^2 \mid \exists \delta^*_{-+}\xi \in L^2 \text{ with } \langle \delta^*_{-+}\xi|\varphi\rangle = \langle \xi|\varphi'\rangle\ \forall \varphi \in \mathrm{dom}(\delta_{-+})\}.$$

Let first $\xi \in \mathrm{dom}(\delta^*_{-+})$. Applying the test functions $\varphi \in C_c^\infty((0,\ell)) \subset \mathrm{dom}(\delta_{-+})$ to the connection $\langle \delta^*_{-+}\xi|\varphi\rangle = \langle \xi|\varphi'\rangle = -\langle \xi'|\varphi\rangle$, we arrive at $\xi \in W^1$ and $\delta^*_{-+}\xi = -\xi'$. In order to prove $\xi' \in \mathrm{dom}(\delta_{-+})$, we integrate partially (extension from smooth functions to W^1 via Proposition 4.2(b))

$$\begin{aligned}
\langle \delta^*_{-+}\xi|\varphi\rangle = -\langle \xi'|\varphi\rangle &= -\int_0^\ell \overline{\xi'(x)}\varphi(x)\,dx\\
&= -\underbrace{\Big[\overline{\xi(x)}\varphi(x)\Big]_0^\ell}_{\text{boundary term}} + \underbrace{\int_0^\ell \overline{\xi(x)}\varphi'(x)\,dx}_{=\langle \xi|\varphi'\rangle}.
\end{aligned}$$

Thus $\langle \delta^*_{-+}\xi|\varphi\rangle = \langle \xi|\varphi'\rangle$ is fulfilled for all $\varphi \in \mathrm{dom}(\delta_{-+})$, if and only if the boundary term vanishes. We know $\varphi(\ell) = 0$ for all $\varphi \in \mathrm{dom}(\delta_{-+}) = \{\varphi \in W^1 \mid \varphi(\ell) = 0\}$, but there is no BV at $x = 0$ for $\varphi \in \mathrm{dom}(\delta_{-+})$. Thus a vanishing boundary term forces $\xi(0) = 0$, implying $\xi \in \mathrm{dom}(\delta_{+-}) = \{\xi \in W^1 \mid \xi(0) = 0\}$. So far, $\delta^*_{-+}\xi = -\xi' = -\delta_{+-}\xi$ for $\xi \in \mathrm{dom}(\delta^*_{-+}) \subseteq \mathrm{dom}(\delta_{+-})$. Conversely, let $\xi \in \mathrm{dom}(\delta_{+-})$. Then the above partial integration yields $\langle -\delta_{+-}\xi|\varphi\rangle = \langle -\xi'|\varphi\rangle = \langle \xi|\varphi'\rangle$ for all $\varphi \in \mathrm{dom}(\delta_{-+})$, implying $\xi \in \mathrm{dom}(\delta^*_{-+})$ and $\delta^*_{-+}\xi = -\xi' = -\delta_{+-}\xi$. Therefore, $\delta^*_{-+} = -\delta_{+-}$, and by adjoining $\delta_{-+} = \delta^{**}_{-+} = -\delta^*_{+-}$. ■

We would like to mention that there exist overcountably many anti-selfadjoint differential operators $\delta_z = -\delta_z^*$ operating in L^2, one for each $z \in \mathbb{C}$ with $|z| = 1$,

$$\delta_z\xi = \xi', \qquad \forall \xi \in \mathrm{dom}(\delta_z) := \{\xi \in W^1 \mid \xi(0) = z\xi(\ell)\},$$

belonging to the boundary condition $\xi(0) = z\xi(\ell)$, e.g. Refs. 10, 11, 15. The δ_z are related to the two types δ_{++} and δ_{--}, in the sense that

$$\delta_{++} \subset \delta_z \subset \delta_{--}, \quad \text{meaning} \quad \mathrm{dom}(\delta_{++}) \subset \mathrm{dom}(\delta_z) \subset \mathrm{dom}(\delta_{--}).$$

δ_{++} is the smallest, δ_{--} the largest differential operator in L^2, whereas all the anti-selfadjoint operators δ_z lie in between, also δ_{+-} and δ_{-+}. When multiplying with $-i$ and Planck's constant $\hbar$, one arrives at the selfadjoint momentum operators $p_z = -i\hbar\delta_z$ used in quantum mechanics on the spatial interval $[0,\ell]$.

4.3. Four different positive, selfadjoint Laplace operators

Recall that the operator product BC of two operators B and C is defined by

$$\mathrm{dom}(BC) = \{\xi \in \mathrm{dom}(C) \mid C\xi \in \mathrm{dom}(B)\}, \quad BC\xi := (BC)\xi = B(C\xi). \tag{4.3}$$

Since for a closed operator B the operator product B^*B is always positive and selfadjoint, one immediately obtains the next result with help of Lemma 4.3. The indices, DN, DD, etc., denote homogeneous Dirichlet or Neumann BV at the left or right end of the beam, respectively.

Corollary 4.4 (*Four Positive, Selfadjoint Laplacians Acting on L^2*). *Consider the following four Laplace operators $\Delta_{..} \in \{\Delta_{DD},\Delta_{NN},\Delta_{DN},\Delta_{ND}\}$ acting on the Hilbert space L^2. It holds $\Delta_{..}\xi = \xi''$ for $\xi \in \mathrm{dom}(\Delta_{..}) \subset W^2$. We multiply with the minus sign in order to obtain positivity.*

(a) The positive, selfadjoint Dirichlet Laplacian $-\Delta_{DD}$ is given by the operator product

$$-\Delta_{DD} = \delta^*_{++}\delta_{++} = -\delta_{--}\delta_{++}.$$

From $\Delta_{DD} = \delta_{--}\delta_{++}$ and the fact that δ_{--} has no BV, it follows that $-\Delta_{DD}$ satisfies the BV for δ_{++}, namely the homogeneous Dirichlet BV $\xi(0) = 0 = \xi(\ell)$. In the literature this boundary condition is indexed by ∞, that means $\Delta_{DD} = \Delta_\infty$.

(b) The positive, selfadjoint Neumann Laplacian $-\Delta_{NN}$ is given by the operator product

$$-\Delta_{NN} = \delta^*_{--}\delta_{--} = -\delta_{++}\delta_{--}.$$

Since δ_{--} has no BV, the BV for $\Delta_{NN} = \delta_{++}\delta_{--}$ arises from the BV of δ_{++} for the derivative $\delta_{--}\xi = \xi'$, namely the homogeneous Neumann BV $\xi'(0) = 0 = \xi'(\ell)$. In the literature this boundary condition is indexed by 0, that is $\Delta_{NN} = \Delta_0$.

(c) The two positive, selfadjoint mixed Laplacians, $-\Delta_{DN}$ and $-\Delta_{ND}$, are given by the operator products

$$-\Delta_{DN} = \delta^*_{+-}\delta_{+-} = -\delta_{-+}\delta_{+-}, \qquad -\Delta_{ND} = \delta^*_{-+}\delta_{-+} = -\delta_{+-}\delta_{-+}$$

with mixed homogeneous Dirichlet and Neumann BV, $\xi(0) = 0 = \xi'(\ell)$ and $\xi'(0) = 0 = \xi(\ell)$, respectively.

Note that by Proposition 4.2(c) we have $W^2 \subseteq C^1([0,\ell])$, and thus ξ and ξ' allow for a boundary evaluation, and so the BV are defined as usual.

The spectra $\sigma(-\Delta_{..})$ of these four Laplacians $-\Delta_{..}$ acting in the Hilbert space L^2 are purely discrete,

$$\sigma(-\Delta_{DD}) = \{\big(\tfrac{n\pi}{\ell}\big)^2 \mid n \in \mathbb{N}\}; \qquad \sigma(-\Delta_{NN}) = \{\big(\tfrac{(n-1)\pi}{\ell}\big)^2 \mid n \in \mathbb{N}\};$$

$$\sigma(-\Delta_{DN}) = \{\big(\tfrac{(2n-1)\pi}{2\ell}\big)^2 \mid n \in \mathbb{N}\}; \quad \sigma(-\Delta_{ND}) = \{\big(\tfrac{(2n-1)\pi}{2\ell}\big)^2 \mid n \in \mathbb{N}\}.$$

The associated normalized eigenfunctions

$$
\begin{aligned}
-\Delta_{DD} &= -\delta_{--}\delta_{++} & \psi_n(x) &= \sqrt{\frac{2}{\ell}}\,\sin(\frac{n\pi}{\ell}x),\ n\in\mathbb{N};\\
-\Delta_{NN} &= -\delta_{++}\delta_{--} & \psi_n(x) &= \sqrt{\frac{2}{\ell}}\,\cos(\frac{(n-1)\pi}{\ell}x),\ n\geq 2,\\
& & \psi_1(x) &= \frac{1}{\sqrt{\ell}},\ n=1;\\
-\Delta_{DN} &= -\delta_{-+}\delta_{+-} & \psi_n(x) &= \sqrt{\frac{2}{\ell}}\,\sin(\frac{(2n-1)\pi}{2\ell}x),\ n\in\mathbb{N};\\
-\Delta_{ND} &= -\delta_{+-}\delta_{-+} & \psi_n(x) &= \sqrt{\frac{2}{\ell}}\,\cos(\frac{(2n-1)\pi}{2\ell}x),\ n\in\mathbb{N};
\end{aligned}
$$

constitute four different ONBs of the Hilbert space $\mathrm{L}^2 = \mathrm{L}^2((0,\ell))$.

4.4. Differential operators of 4th order for beam BV from statics

In statics usually the following three support possibilities are used. That is, for the transverse, purely spatial bending function $\xi(x)$ we choose at one end $x^\star$ of the beam, $x^\star = 0$ or $x^\star = \ell$, different BV, e.g. Ref. 2 etc.,

$$
\begin{aligned}
&\text{(a) = flexible support} && \xi(x^\star)=0\,, && \xi''(x^\star)=0\,;\\
&\text{(b) = fixed support} && \xi(x^\star)=0\,, && \xi'(x^\star)=0\,;\\
&\text{(c) = free end} && \xi''(x^\star)=0\,, && \xi'''(x^\star)=0\,.
\end{aligned}
$$

Especially, flexible support (a) allows rotations with no moment resistance, and free end (c) in addition translations, see Theorem 4.8(c) below.

Notational Remark 4.5. If the left end of the beam is supported according to (a) and the right end by (b), then we briefly call the beam to be (a)–(b) supported. Analogously, (c)–(b), (b)–(a), …, and so on.

For these different possibilities of support at the ends of the beam we will construct positive, selfadjoint differential operators A of 4th order, $A\xi = \xi^{(4)} = \xi''''$, so that the BV are respected. Moreover, we add three cases, which are not in agreement with the above supports known from engineering statics.

Subsequently we list in the first column the considered support of the beam, then the associated four BV, and finally in the third column the corresponding *product* differential operator $\hat{A}$ of 4th order respecting exactly these four support BV (see (4.3) for operator products). The positive, selfadjoint A will turn out to be a unique extension of the product operator $\hat{A}$.

(a)–(a)	BV: $\xi(0)=0,\ \xi(\ell)=0,\ \xi''(0)=0,\ \xi''(\ell)=0,$	$\hat{A}=\delta_{--}\delta_{++}\delta_{--}\delta_{++};$
(a)–(b)	BV: $\xi(0)=0,\ \xi(\ell)=0,\ \xi'(\ell)=0,\ \xi''(0)=0,$	$\hat{A}=\delta_{--}\delta_{+-}\delta_{-+}\delta_{++};$
(a)–(c)	BV: $\xi(0)=0,\ \xi''(0)=0,\ \xi''(\ell)=0,\ \xi'''(\ell)=0,$	$\hat{A}=\delta_{-+}\delta_{++}\delta_{--}\delta_{+-};$
(b)–(b)	BV: $\xi(0)=0,\ \xi(\ell)=0,\ \xi'(0)=0,\ \xi'(\ell)=0,$	$\hat{A}=\delta_{--}\delta_{--}\delta_{++}\delta_{++};$
(b)–(c)	BV: $\xi(0)=0,\ \xi'(0)=0,\ \xi''(\ell)=0,\ \xi'''(\ell)=0,$	$\hat{A}=\delta_{-+}\delta_{-+}\delta_{+-}\delta_{+-};$
(c)–(c)	BV: $\xi''(0)=0,\ \xi''(\ell)=0,\ \xi'''(0)=0,\ \xi'''(\ell)=0,$	$\hat{A}=\delta_{++}\delta_{++}\delta_{--}\delta_{--};$
add-(i)	BV: $\xi'(0)=0,\ \xi'(\ell)=0,\ \xi'''(0)=0,\ \xi'''(\ell)=0,$	$\hat{A}=\delta_{++}\delta_{--}\delta_{++}\delta_{--};$
add-(ii)	BV: $\xi(0)=0,\ \xi'(\ell)=0,\ \xi''(0)=0,\ \xi'''(\ell)=0,$	$\hat{A}=\delta_{-+}\delta_{+-}\delta_{-+}\delta_{+-};$
add-(iii)	BV: $\xi(\ell)=0,\ \xi'(0)=0,\ \xi''(\ell)=0,\ \xi'''(0)=0,$	$\hat{A}=\delta_{+-}\delta_{-+}\delta_{+-}\delta_{-+}.$

For the remaining possibilities (b)–(a), (c)–(a), (c)–(b) of statics, simply invert the beam. Also BV add-(iii) is the inverted beam with BV add-(ii). We have $\mathrm{C}_c^\infty((0,\ell)) \subset \mathrm{dom}(\hat{A}) \subset \mathrm{W}^4$, hence each $\hat{A}$ is densely defined in L^2. Note that

$$\hat{A}\xi = \xi'''', \quad \forall \xi \in \mathrm{dom}(\hat{A}) = \{\xi \in \mathrm{W}^4 \mid \xi \text{ fulfills all 4 BV of } \hat{A}\}.$$

For an operator A to be an extension of the operator $\hat{A}$, means

$$\mathrm{dom}(\hat{A}) \subseteq \mathrm{dom}(A) \qquad \text{with} \qquad \hat{A}\xi = A\xi\,,\ \ \forall \xi \in \mathrm{dom}(\hat{A}),$$

denoted as $\hat{A} \subseteq A$. We write $\hat{A} \subset A$ or equivalently $A \supset \hat{A}$, if A is a *genuine* operator extension of $\hat{A}$, that is $\mathrm{dom}(\hat{A}) \subsetneq \mathrm{dom}(A)$.

Theorem 4.6 (*Existence of Unique Positive, Selfadjoint Extensions*)**.** *In each of the above cases, there exists a unique positive, selfadjoint extension $A \supseteq \hat{A}$ for the product operator $\hat{A}$ such that $\mathrm{dom}(A)$ is a subset of*

$$\{\xi \in \mathrm{W}^2 \mid \xi \text{ fulfills the BV for } \xi \text{ and } \xi' \text{ of } \hat{A} \text{ but not for higher derivatives}\}.$$

For that unique A it holds

$$A\xi = \hat{A}\xi = \xi'''' \qquad \text{for all} \quad \xi \in \mathrm{dom}(A) \cap \mathrm{W}^4 = \mathrm{dom}(\hat{A}).$$

The proof is given in Section 6.

A is the so called *Friedrichs extension* of the product operator $\hat{A}$, the smallest form extension of $\hat{A}$, e.g. Refs. 9–11, 13. Possibly there may exist further positive, selfadjoint extensions of $\hat{A}$, but their domains of definition contain elements not from
$\{\xi \in \mathrm{W}^2 \mid \xi$ fulfills BV for ξ, ξ' of $\hat{A}$ but not for higher derivatives$\}$.

Corollary 4.7. *We distinguish two groups of operators of type A or $\hat{A}$:*

(I) In each of the cases (a)–(a), add-(i), add-(ii), and add-(iii), the positive, selfadjoint A is not a genuine operator extension of $\hat{A}$, since already the product operator $\hat{A}$ is positive and selfadjoint, and thus coincides with A,

$$
\begin{aligned}
&\textit{(a)–(a)} && A=\hat{A}=\delta_{--}\delta_{++}\delta_{--}\delta_{++}=(-\Delta_{DD})^2;\\
&\textit{add-(i)} && A=\hat{A}=\delta_{++}\delta_{--}\delta_{++}\delta_{--}=(-\Delta_{NN})^2;\\
&\textit{add-(ii)} && A=\hat{A}=\delta_{-+}\delta_{+-}\delta_{-+}\delta_{+-}=(-\Delta_{DN})^2;\\
&\textit{add-(iii)} && A=\hat{A}=\delta_{+-}\delta_{-+}\delta_{+-}\delta_{-+}=(-\Delta_{ND})^2.
\end{aligned}
$$

(II) In the cases (a)–(b), (a)–(c), (b)–(b), (b)–(c), (c)–(c), the positive, selfadjoint extension A is a genuine operator extension of the original product operator $\hat{A}$, that is $A \supset \hat{A}$. Here A is defined in terms of a positive closed sesquilinear form. Therefore, neither the spectrum of A nor that of any of the sandwiched operator products $\hat{A}$, like $\hat{A} = \delta_{-+}\underbrace{\delta_{++}\delta_{--}}_{=\Delta_{NN}}\delta_{+-}$ for (a)–(c) support, are related to the spectra of the Laplacians from Corollary 4.4.

Proof. The result for (I) is a consequence of Section 4.3, since the Laplacians are operator products. So $\hat{A}$ is already selfadjoint, and for a selfadjoint operator there do not exist selfadjoint extensions. (II) is obvious. ■

Since $\Lambda = (0,\ell)$ is interior, one obtains a spectral result as in Section 3.4.

Theorem 4.8. *For each positive, selfadjoint operator A of Theorem 4.6 it holds:*

(a) A has a pure point (= purely discrete) spectrum $\sigma(A) = \sigma_p(A) \subset [0,\infty)$.
(b) Each eigenspace is finite dimensional and the eigenvalues a_n, $n \in \mathbb{N}$, may be arranged increasingly so that (3.6) is satisfied.
(c) Only the cases (a)–(c), (c)–(c), and add-(i) possess the eigenvalue zero.

Part (c) means that rotation of the beam is allowed, and for (c)–(c) even translation is possible. This corresponds to eigenfunctions of type $\eta(x) = a + bx$ for all $x \in (0,\ell)$ with constants $a, b \in \mathbb{R}$, possessing eigenvalue zero since $\eta'' = 0$. For the proof of Theorem 4.8 see Section 6.

4.5. Euler–Bernoulli differential equation in $\mathrm{L}^2((0,\ell))$ for the beam

The Euler–Bernoulli differential equation IBVP for the bending vibrations of the beam is written in L^2–Hilbert space language as in

Theorem 2.1, namely

$$\text{differential equation} \quad \frac{d^2u(t)}{dt^2} = -\varsigma^2 A u(t)\,, \quad t \in \mathbb{R}\,, \tag{4.4}$$

$$\text{IV (at } t=0) \quad u(t)|_{t=0} = u_0 \in L^2, \quad \left.\frac{du(t)}{dt}\right|_{t=0} = \dot{u}_0 \in L^2,$$

with given IV $u_0,\ \dot{u}_0 \in L^2$. The positive, selfadjoint differential (extension) operator $A \supseteq \hat{A}$ of 4th order has to be chosen according to Theorem 4.6 for the considered support, (a)–(b), (b)–(b), etc., multiplied with the material constant $\varsigma^2 := \frac{EI}{\rho A_{cs}}$ arising from the constant cross-sectional area A_{cs}, the mass density per unit length ρ, the elasticity modulus E for the material of the beam, and the second area moment I of the cross-section, e.g. Ref. 2, etc.

With the purely discrete spectrum $\{a_n \geq 0 \mid n \in \mathbb{N}\}$ of A and corresponding normalized eigenvectors $\psi_n \in L^2$, $n \in \mathbb{N}$, constituting an ONB, the solution of the IBVP (4.4) is given with (2.6) by the spectral decomposition

$$u(t) = \sum_{n=1}^{\infty}\Big(\cos(t\varsigma\sqrt{a_n})\langle\psi_n|u_0\rangle + \frac{\sin(t\varsigma\sqrt{a_n})}{\varsigma\sqrt{a_n}}\langle\psi_n|\dot{u}_0\rangle\Big)\psi_n\,, \quad t \in \mathbb{R}\,. \tag{4.5}$$

Recall, the operator A is $\frac{d^4}{dx^4}$ with a specific domain of definition consisting of smooth functions satisfying the BV (regularity). Therefore, the eigenequation for an eigenfunction ψ_n is just the ordinary differential equation of 4th order,

$$A\psi_n(x) = \psi_n''''(x) = a_n\psi_n(x)\,, \qquad \forall x \in (0,\ell)\,, \qquad a_n \geq 0\,, \tag{4.6}$$

where e.g. the support (a)–(b) causes the BV $\psi_n(0) = 0, \psi_n(\ell) = 0, \psi_n'(\ell) = 0, \psi_n''(0) = 0$.

Summary 4.9. *The eigenequation BVP (4.6) is analytically solvable only for BV (a)–(a), add-(i), add-(ii), and add-(iii) of group (I) in Corollary 4.7. For group (II) the eigenequation (4.6) is solvable numerically, only, cf. Ref. 2 and references therein.*

4.6. Euler–Bernoulli for BV (a)–(a), add-(i), add-(ii), add-(iii) of group (I)

In the four BV cases of Corollary 4.7(I) the eigenequation (4.6) is solvable analytically. Since each positive, selfadjoint $A = \hat{A}$ is the square of one of the four Laplacians in Corollary 4.4, the eigenspectrum $\{a_n \mid n \in \mathbb{N}\}$ of $A = (-\Delta_{..})^2$ is the square of the eigenvalues of the Laplacians, respectively, however with the same normalized eigenvectorfunctions ψ_n, $n \in \mathbb{N}$, as in Section 4.3,

$$\text{(a)–(a)} \quad A = (-\Delta_{DD})^2: \quad a_n = \big(\tfrac{n\pi}{\ell}\big)^4, \qquad \psi_n(x) = \sqrt{\tfrac{2}{\ell}}\,\sin(\tfrac{n\pi}{\ell}x)\,;$$

$$\text{add-(i)} \quad A = (-\Delta_{NN})^2: \quad a_n = \big(\tfrac{(n-1)\pi}{\ell}\big)^4, \quad \psi_n(x) \overset{n\neq 1}{=} \sqrt{\tfrac{2}{\ell}}\,\cos(\tfrac{(n-1)\pi}{\ell}x)\,;$$

$$\text{add-(ii)} \quad A = (-\Delta_{DN})^2: \quad a_n = \big(\tfrac{(2n-1)\pi}{2\ell}\big)^4, \quad \psi_n(x) = \sqrt{\tfrac{2}{\ell}}\,\sin(\tfrac{(2n-1)\pi}{2\ell}x)\,;$$

$$\text{add-(iii)} \quad A = (-\Delta_{ND})^2: \quad a_n = \big(\tfrac{(2n-1)\pi}{2\ell}\big)^4, \quad \psi_n(x) = \sqrt{\tfrac{2}{\ell}}\,\cos(\tfrac{(2n-1)\pi}{2\ell}x)\,.$$

In each case the unique solution of the Euler–Bernoulli IBVP (4.4) is given by Eq. (4.5). For add-(i) with $A = (-\Delta_{NN})^2$ we have for $n = 1$ the eigenvalue $a_1 = 0$ with eigenfunction $\psi_1(x) = \frac{1}{\sqrt{\ell}}$, and it holds $\cos(t\varsigma\sqrt{a_1}) \overset{a_1=0}{=} \cos(0) = 1$ and $\frac{\sin(t\varsigma\sqrt{a_1})}{\varsigma\sqrt{a_1}} \overset{a_1=0}{=} t$ in accordance with Eq. (2.4).

5. Bending vibrations of a beam versus wave swinging of a string

Here we compare the solution for the beam bending equation derived in the prior section with the solution of a wave equation. Directly comparable are the support case (a)–(a) for the Euler–Bernoulli IBVP and homogeneous Dirichlet BV for the wave IBVP, only.

5.1. Bending vibrations with (a)–(a) support of the beam

We take the (a)–(a) operator $A = (-\Delta_{DD})^2$ for the **Euler–Bernoulli IBVP** from (4.4). The eigenquantities for $A = (-\Delta_{DD})^2$ are stated in Section 4.6. The inner products of the eigenvectors ψ_n with the IV $u_0, \dot{u}_0$,

$$S_n := \sqrt{\frac{2}{\ell}}\langle\psi_n|u_0\rangle\,, \qquad \dot{S}_n := \sqrt{\frac{2}{\ell}}\langle\psi_n|\dot{u}_0\rangle\,, \tag{5.1}$$

appearing in (4.5), constitute just the sine Fourier coefficients S_n and $\dot{S}_n$ of the (odd extensions of the) IV functions $u_0,\ \dot{u}_0 \in L^2$, respectively. Then the unique solution (4.5) of the Euler–Bernoulli IBVP is the Fourier series expansion

$$u(t)(x) = u(x,t) = \sum_{n=1}^{\infty}\Big[S_n\cos(\omega_n t) + \frac{\dot{S}_n}{\omega_n}\sin(\omega_n t)\Big]\sin(\frac{n\pi}{\ell}x) \tag{5.2}$$

for all $x \in [0,\ell]$ and all $t \in \mathbb{R}$, where $\omega_n := \varsigma\sqrt{a_n} = \big(\frac{n\pi}{\ell}\big)^2\varsigma$ for each $n \in \mathbb{N}$.

For completeness we state the Euler–Bernoulli (E-B) IBVP in function language with ordinary solution function $u(t)(x) = u(x,t)$ and support (a)–(a),

$$\begin{array}{lll}
\text{E-B PDE} & \partial_t^2 u = -\varsigma^2\partial_x^4 u\,, & x \in (0,\ell)\,,\ t \in \mathbb{R}\,,\\
\text{IV (at } t=0) & u(x,0) = u_0(x)\,, \quad \partial_t u(x,0) = \dot{u}_0(x)\,, & x \in (0,\ell)\,,\\
\text{BV} & u(0,t) = u(\ell,t) = \partial_x^2 u(0,t) = \partial_x^2 u(\ell,t) = 0\,, & t \in \mathbb{R}\,.
\end{array}$$

5.2. Wave swinging with homogeneous Dirichlet BV of the string

The IBVP for the **wave equation** in one spatial dimension with homogeneous Dirichlet BV is written in function language as

$$\begin{array}{lll}
\text{wave PDE} & \partial_t^2 u = \varsigma^2\partial_x^2 u\,, & x \in (0,\ell)\,, \quad t \in \mathbb{R}\,,\\
\text{IV (at } t=0) & u(x,0) = u_0(x)\,, \quad \partial_t u(x,0) = \dot{u}_0(x)\,, & x \in (0,\ell)\,,\\
\text{BV} & u(0,t) = u(\ell,t) = 0\,, & t \in \mathbb{R}\,,
\end{array}$$

here $\varsigma > 0$ is the wave speed, and $u_0(x), \dot{u}_0(x)$ are two given IV functions. The solution can be interpreted as a vibrating string (of a violin or guitar) with length $\ell > 0$, which is fixed at both ends. In Hilbert space language we have

$$\text{wave equation} \quad \frac{d^2u(t)}{dt^2} = -\varsigma^2\overbrace{(-\Delta_{DD})}^{=A}u(t)\,, \quad t \in \mathbb{R}\,,$$

$$\text{IV (at } t=0) \quad u(t)|_{t=0} = u_0 \in L^2\,, \quad \left.\frac{du(t)}{dt}\right|_{t=0} = \dot{u}_0 \in L^2\,,$$

with given IV $u_0, \dot{u}_0 \in L^2$. The BV are covered by the positive, selfadjoint Dirichlet Laplacian $A = -\Delta_{DD}$ from Corollary 4.4. The L^2–solution trajectory is given by (4.5) but here with the eigenvalues $a_n = \big(\frac{n\pi}{\ell}\big)^2$ for $A = -\Delta_{DD}$ and eigenfunctions $\psi_n(x) = \sqrt{\frac{2}{\ell}}\sin(\frac{n\pi}{\ell}x)$ as before, cf. Section 4.3. With the Fourier coefficients S_n and $\dot{S}_n$ of Eq. (5.1) the solution is written like the Fourier series expansion (5.2), however now with $\omega_n := \varsigma\sqrt{a_n} = \frac{n\pi}{\ell}\varsigma$, $n \in \mathbb{N}$.

Summary 5.1. *The solutions, Euler–Bernoulli and wave, differ only in the exponent $j = 2$ or $j = 1$ for the associated selfadjoint, positive differential operators $A = (-\Delta_{DD})^j$.*

6. Proof of Theorems 4.6 and 4.8 from Section 4.4

Part (a). By Ref. 12 theorem 2.6(2) there is a $c > 0$ so that ($\|.\|$ is the L^2–norm)

$$\|\xi'\| \leq c\big(\|\xi\| + \|\xi''\|\big)\,, \quad \forall \xi \in W^2.$$

Consequently, we arrive at the following estimate for the second Sobolev norm $\|.\|_2$

$$\begin{aligned}\|\xi\|_2^2 &= \|\xi\|^2 + \|\xi'\|^2 + \|\xi''\|^2 \le \|\xi\|^2 + c^2\big(\|\xi\| + \|\xi''\|\big)^2 + \|\xi''\|^2 \\ &= (1+c^2)\|\xi\|^2 + (1+c^2)\|\xi''\|^2 + 2c^2\|\xi\|\|\xi''\| \\ &\overset{\star}{\le} (1+c^2)\|\xi\|^2 + (1+c^2)\|\xi''\|^2 + c^2\big(\|\xi\|^2 + \|\xi''\|^2\big) \\ &= (1+2c^2)\underbrace{\big(\|\xi\|^2 + \|\xi''\|^2\big)}_{=:\ \|\xi\|_s^2} \le (1+2c^2)\underbrace{\big(\|\xi\|^2 + \|\xi'\|^2 + \|\xi''\|^2\big)}_{=\ \|\xi\|_2^2}.\end{aligned}$$

At $\overset{\star}{\le}$ we used $0 \le (a-b)^2 = a^2 + b^2 - 2ab$, therefore $2ab \le a^2 + b^2$. That means, the norm $\|.\|_s$ and the second Sobolev norm $\|.\|_2$ are equivalent on W^2.

For the demonstration how to proceed, let us take for example the support case (a)–(b); the other cases work analogously. For (a)–(b) it is $\hat{A} = \delta_{--}\delta_{+-}\delta_{-+}\delta_{++}$. Taking adjoints according to Lemma 4.3 one gets

$$\langle \xi | \hat{A}\eta \rangle = \langle \delta_{-+}\delta_{++}\xi | \delta_{-+}\delta_{++}\eta \rangle, \quad \forall \xi \in \mathrm{dom}(\delta_{-+}\delta_{++}), \quad \forall \eta \in \mathrm{dom}(\hat{A}) \subseteq \mathrm{W}^4.$$

We define the positive sesquilinear form s by

$$s(\xi,\eta) := \langle \delta_{-+}\delta_{++}\xi | \delta_{-+}\delta_{++}\eta \rangle,$$

for $\xi,\eta \in \mathrm{dom}(s) := \mathrm{dom}(\delta_{-+}\delta_{++}) = \{\eta \in \mathrm{W}^2 \mid \eta(0)=0,\ \eta(\ell)=0,\ \eta'(\ell)=0\}$ (since $\mathrm{W}^2 \subseteq \mathrm{C}^1([0,\ell])$ the boundary terms are well defined). Because of the equivalence of $\|.\|_s$ and the second Sobolev norm $\|.\|_2$, it follows that the form s is closed, since $\{\xi \in \mathrm{W}^2 \mid \xi(0)=0,\ \xi(\ell)=0,\ \xi'(\ell)=0\}$ is a $\|.\|_2$–closed subspace of the Sobolev space W^2. Especially, the product operator $\delta_{-+}\delta_{++}$ is closed.

We now cite[9] subsection VI § 2, 1 with a result, which is valid for every positive closed form in any real or complex Hilbert space $\mathcal{H}$ (here $\mathcal{H} = \mathrm{L}^2$).

Theorem 6.1 (*First Representation Theorem*[9]). *For the positive closed form s there exists a positive, selfadjoint operator A acting on $\mathcal{H}$, such that:*

(i) $\mathrm{dom}(A) \subseteq \mathrm{dom}(s)$, *and* $s(\xi,\eta) = \langle \xi | A\eta \rangle$, $\forall \xi \in \mathrm{dom}(s)$, $\forall \eta \in \mathrm{dom}(A)$.
(ii) $\mathrm{dom}(A)$ *is a form core for s.*
(iii) If for $\eta \in \mathrm{dom}(s)$ and $\varphi \in \mathcal{H}$ it holds $s(\xi,\eta) = \langle \xi|\varphi \rangle$ for all ξ from a form core of s, then $\eta \in \mathrm{dom}(A)$ and $A\eta = \varphi$.

Moreover, uniqueness of A is given by (i).

Suppose $\xi \in \mathrm{W}^2$ and $\eta \in \mathrm{W}^4$. Then two times integrating partially leads to (extension from smooth functions by Proposition 4.2(b))

$$\langle \xi'' | \eta'' \rangle = \Big[\overline{\xi'}\eta''\Big]_0^\ell - \Big[\overline{\xi}\eta'''\Big]_0^\ell + \langle \xi | \eta'''' \rangle. \tag{6.1}$$

Recall, $\mathrm{W}^m \subseteq \mathrm{C}^k([0,\ell])$ for $m > k$ and so the boundary terms are well defined. Inserting $\xi \in \mathrm{dom}(s)$ and $\eta \in \mathrm{W}^4 \cap \mathrm{dom}(s)$ in (6.1), we arrive at

$$\langle \xi'' | \eta'' \rangle = -\overline{\xi'(0)}\eta''(0) + \langle \xi | \eta'''' \rangle.$$

Consequently we arrive at the equivalence

$$s(\xi,\eta) = \langle \xi'' | \eta'' \rangle = \langle \xi | \eta'''' \rangle \ \ \forall \xi \in \mathrm{dom}(s) \qquad \Leftrightarrow \qquad \eta''(0) = 0.$$

For $\eta \in \mathrm{W}^4 \cap \mathrm{dom}(s)$ satisfying $\eta''(0)=0$ we conclude from (iii) of the first representation theorem that $\eta \in \mathrm{dom}(A)$ and $A\eta = \eta''''$. In other words,

$$\mathrm{dom}(\hat{A}) = \{\eta \in \mathrm{W}^4 \mid \eta(0)=0,\ \eta(\ell)=0,\ \eta'(\ell)=0,\ \eta''(0)=0\} \subseteq \mathrm{dom}(A),$$

and consequently, A is an extension of $\hat{A} = \delta_{--}\delta_{+-}\delta_{-+}\delta_{++}$.

If conversely, $\eta \in \mathrm{dom}(A) \cap \mathrm{W}^4$, then doubled partial integration for all $\xi \in \mathrm{dom}(s)$ compared with (i) of the first representation Theorem 6.1 ensures $\eta''(0)=0$. Thus $\mathrm{dom}(\hat{A}) = \mathrm{dom}(A) \cap \mathrm{W}^4$.

From example 2.13 of Ref. 9 subsection VI § 2, 4 one concludes that $A = (\delta_{-+}\delta_{++})^*\delta_{-+}\delta_{++}$.

So far we have proven Theorem 4.6 up to the stated uniqueness.

PART (B). Let $\hat{A}$ be one support case. First note that the above form s is a closed extension of the positive form

$$\hat{s}(\xi,\eta) := \langle \xi | \hat{A}\eta \rangle, \quad \xi,\eta \in \mathrm{dom}(\hat{s}) := \mathrm{dom}(\hat{A}).$$

For the proof that s is indeed the smallest closed extension of $\hat{s}$ see PART (D). The heuristics behind that is given here: First remember, the norm $\|.\|_s$ is equivalent to the second Sobolev norm $\|.\|_2$ by PART (A), so the $\|.\|_s$–closure of $\mathrm{dom}(\hat{s})$ coincides with its closure with respect to $\|.\|_2$ within W^2. Now take into account the fact that $\xi \in \mathrm{W}^2$ does not possess a boundary evaluation for ξ'' and ξ''', only for ξ and ξ'. So, when performing the $\|.\|_2$–closure of $\mathrm{dom}(\hat{s}) = \mathrm{dom}(\hat{A})$ within W^2, the BV for ξ'' and ξ''' of $\xi \in \mathrm{dom}(\hat{A})$ are no longer respected, and the $\|.\|_2$–closure of $\mathrm{dom}(\hat{A})$ should agree with

$$\mathrm{dom}(s) = \{\xi \in \mathrm{W}^2 \mid \xi \text{ fulfills BV for } \xi,\xi' \text{ of } \hat{A}, \text{ not for higher derivat.}\}.$$

Consequently, s is the smallest closed extension, i.e. the closure, of the positive form $\hat{s}$, and A is the Friedrichs extension of $\hat{A}$,[9] subsection VI § 2, 3.

For the proof of uniqueness of A, assume that $\check{A} \supseteq \hat{A}$ is a positive, selfadjoint extension of $\hat{A}$. Then the corresponding positive form $\check{s}(\xi,\eta) = \langle \xi | \check{A}\eta \rangle$, $\xi,\eta \in \mathrm{dom}(\check{s}) := \mathrm{dom}(\check{A})$, is closable, its closure be denoted by the same symbol $\check{s}$. The operator $\check{A}$ is the operator associated to $\check{s}$ by the first representation theorem, corollary 2.2 of Ref. 9 subsection VI § 2, 1, that is,

$$\check{s}(\xi,\eta) = \langle \xi | \check{A}\eta \rangle, \quad \forall \xi \in \mathrm{dom}(\check{s}), \quad \forall \eta \in \mathrm{dom}(\check{A}) \subseteq \mathrm{dom}(\check{s}).$$

Since s is the smallest closed form extending $\hat{s}$, one concludes $\hat{s} \subseteq s \subseteq \check{s}$. If $\mathrm{dom}(\check{A}) \subseteq \mathrm{dom}(s) \subseteq \mathrm{dom}(\check{s})$, then by restriction to $\mathrm{dom}(s)$,

$$s(\xi,\eta) = \check{s}(\xi,\eta) = \langle \xi | \check{A}\eta \rangle, \quad \forall \xi \in \mathrm{dom}(s), \quad \forall \eta \in \mathrm{dom}(\check{A}) \subseteq \mathrm{dom}(s).$$

According to the uniqueness stated in (i) in the first representation theorem, it follows $\check{A} = A$ and $\check{s} = s$. So, A is the unique positive, selfadjoint extension of $\hat{A}$, which fulfills the stated property $\mathrm{dom}(A) \subseteq \mathrm{dom}(s)$.

PART (C). We prove here Theorem 4.8. The identical embeddings $\mathrm{W}^2 \hookrightarrow \mathrm{W}^1 \hookrightarrow \mathrm{L}^2$ are compact by Proposition 4.2(d). The equivalence of norms ensures $(\mathrm{dom}(s), \|.\|_s) \hookrightarrow \mathrm{L}^2$ to be compact. So (a) and (b) of Theorem 4.8 follow from Ref. 15 proposition 43.5-11, a result outlined also in many further textbooks.

We turn to Theorem 4.8(c). η contained in the kernel of A means $\eta \in \mathrm{dom}(A)$ with $A\eta = 0$ (kernel = eigenspace to eigenvalue zero), which leads to $0 = \langle \eta | A\eta \rangle = s(\eta,\eta) = \|\eta''\|^2$. Thus $\eta'' = 0$. This is a vanishing second distributional derivative, so we may conclude $\eta(x) = a + bx$. Up to (a)–(c), (c)–(c), and add-(i), the other support BV possibilities imply $a = b = 0$ and consequently $\eta = 0$. Inserting the BV (a)–(c), (c)–(c), or add-(i) into $\eta(x) = a + bx$ finally proves (c). For (a)–(c) and add-(i) the eigenspace to the eigenvalue zero is one-dimensional, whereas for (c)–(c) the kernel is two-dimensional.

PART (D). We use the Poincaré estimate, see Ref. 12 section 2.3,

$$\|\xi\| \le k\big(\|\xi'\| + |\langle 1|\xi \rangle|\big), \quad \forall \xi \in \mathrm{W}^1, \tag{6.2}$$

with a constant $k > 0$. Here $\langle 1|\xi \rangle = \int_0^\ell \xi(x)\,dx$ is the inner product of ξ with the constant unit function $1(x) = 1$ for all $x \in (0,\ell)$. Applying (6.2) to ξ' yields

$$\|\xi'\| \le k\big(\|\xi''\| + |\langle 1|\xi' \rangle|\big), \quad \forall \xi \in \mathrm{W}^2. \tag{6.3}$$

Recall, $\xi \in \mathrm{W}^2 \subseteq \mathrm{C}^1([0,\ell])$ is continuously differentiable, and ξ'' is defined in the distributional sense. Then for all $\xi \in \mathrm{W}^2$,

$$\begin{aligned}\|\xi\|_s^2 &= \|\xi\|^2 + \|\xi''\|^2 \overset{(6.2)}{\le} k^2\big[\|\xi'\| + |\langle 1|\xi \rangle|\big]^2 + \|\xi''\|^2 \\ &\overset{(6.3)}{\le} k^2\big[k\big(\|\xi''\| + |\langle 1|\xi' \rangle|\big) + |\langle 1|\xi \rangle|\big]^2 + \|\xi''\|^2 \\ &\le \cdots \le d\big(\|\xi\|^2 + \|\xi'\|^2 + \|\xi''\|^2\big) = d\|\xi\|_2^2\end{aligned}$$

with some constant $d > 0$. For the latter inequality one has to use the estimate $|\langle 1|\eta\rangle| \le \|1\|\|\eta\|$ and inequalities like $2ab \le a^2+b^2$ as in PART (A). Thus the norm $\|\xi\|_t^2 := \|\xi''\|^2 + |\langle 1|\xi'\rangle|^2 + |\langle 1|\xi\rangle|^2$ is a third norm on W^2 being equivalent to $\|.\|_s$ and the second Sobolev norm $\|.\|_2$. The associated inner product reads

$$\langle\xi|\eta\rangle_t = \langle\xi''|\eta''\rangle + \langle\xi'|1\rangle\langle 1|\eta'\rangle + \langle\xi|1\rangle\langle 1|\eta\rangle, \quad \forall\xi,\eta\in W^2.$$

Suppose that the closure $\bar{\hat{s}}$ of the form $\hat{s}(\xi,\eta) = \langle\xi|\hat{A}\eta\rangle$, $\xi,\eta \in \mathrm{dom}(\hat{s}) = \mathrm{dom}(\hat{A})$, from PART (B) does not agree with the closed, positive form s from PART (A). That means, we have the proper form inclusion $\bar{\hat{s}} \subset s$, or equivalently, $\mathrm{dom}(\bar{\hat{s}})$ is a proper closed subspace of $\mathrm{dom}(s)$ with respect to the equivalent norms $\|.\|_t \sim \|.\|_s \sim \|.\|_2$ on W^2. Then there exists a $\vartheta \in \mathrm{dom}(s)$, which is orthogonal to $\mathrm{dom}(\bar{\hat{s}})$ with respect to the inner product $\langle .|.\rangle_t$, meaning

$$0 = \langle\xi|\vartheta\rangle_t = \langle\xi''|\vartheta''\rangle + \langle\xi'|1\rangle\langle 1|\vartheta'\rangle + \langle\xi|1\rangle\langle 1|\vartheta\rangle, \tag{6.4}$$

for all $\xi \in \mathrm{dom}(\bar{\hat{s}})$, or equivalently, for all ξ from its form core $\mathrm{dom}(\hat{s}) = \mathrm{dom}(\hat{A})$.

Lemma 6.2. *Let $\phi \in L^2$. The following assertions are valid:*

(a) If $0 = \langle\xi'|\phi\rangle$ for all $\xi \in C_c^\infty((0,\ell))$, then $\phi = a$ in $(0,\ell)$ with an $a \in \mathbb{C}$.
(b) Suppose a $\beta \in \mathbb{C}$ such that $0 = \langle\xi''|\phi\rangle - 2\beta\langle\xi|1\rangle$ for all $\xi \in C_c^\infty((0,\ell))$. Then there exist constants $a,b \in \mathbb{C}$ with $\phi = a+bx+\beta x^2$ in $(0,\ell)$.

In the context of distribution theory, (a) is well known as Hilbert's lemma.

Proof. Fix a $\varphi_0 \in C_c^\infty((0,\ell))$ with $-1 = \langle 1|\varphi_0\rangle = \int_0^\ell \varphi_0(y)\,dy$. For each $\xi \in C_c^\infty((0,\ell))$ define $\psi(x) := \int_0^x \big(\xi(y)+\langle 1|\xi\rangle\varphi_0(y)\big)dy$ for $x \in (0,\ell)$. Then $\psi \in C_c^\infty((0,\ell))$ with compact support contained in $\mathrm{supp}(\varphi_0) \cup \mathrm{supp}(\xi)$. It holds $\psi'(x) = \xi(x) + \langle 1|\xi\rangle\varphi_0(x)$ and $\psi''(x) = \xi'(x) + \langle 1|\xi\rangle\varphi_0'(x)$ for all $x \in (0,\ell)$.
(a) Inserting $\psi \in C_c^\infty((0,\ell))$ with $a := -\langle\varphi_0|\phi\rangle$ yields $0 = \langle\psi'|\phi\rangle = \langle\xi|\phi - a\rangle$. ξ may be chosen arbitrarily and $C_c^\infty((0,\ell))$ is dense in L^2, thus $\phi - a = 0$.
(b) Double partial integration (PI) leads to $0 = \langle\xi''|\phi\rangle - 2\beta\langle\xi|1\rangle \overset{PI}{=} \langle\xi''|\phi - \beta x^2\rangle$. Inserting $\psi \in C_c^\infty((0,\ell))$ and defining $\tilde{\phi} := \phi - \beta x^2$ yields

$$\begin{aligned}0 = \langle\psi''|\tilde{\phi}\rangle &= \langle\xi'|\tilde{\phi}\rangle + \langle\xi|1\rangle\langle\varphi_0'|\tilde{\phi}\rangle = \langle\xi'|\tilde{\phi}\rangle + \langle\xi|b\rangle \overset{PI}{=} \langle\xi'|\tilde{\phi}\rangle - \langle\xi'|bx\rangle \\ &= \langle\xi'|\tilde{\phi} - bx\rangle, \quad \text{where } b := \langle\varphi_0'|\tilde{\phi}\rangle\end{aligned}$$

So by (a) it follows $\tilde{\phi} - bx = a$, so $\phi = a + bx + \beta x^2$. ∎

Restricting the orthogonality (6.4) to test functions $\xi \in C_c^\infty((0,\ell))$, we get

$$0 = \langle\xi''|\vartheta''\rangle + \langle 1|\vartheta'\rangle \underbrace{\langle\xi'|1\rangle}_{=0} + \langle 1|\vartheta\rangle\langle\xi|1\rangle = \langle\xi''|\vartheta''\rangle + \underbrace{\langle 1|\vartheta\rangle}_{-2\beta}\langle\xi|1\rangle,$$

since $\langle 1|\xi'\rangle = \int_0^\vartheta \xi'(x)\,dx = \xi(\ell) - \xi(0) = 0$ because of the compact support of ξ. Then part (b) of the previous lemma implies

$$\vartheta'' = a + bx - \frac{\langle 1|\vartheta\rangle}{2}x^2, \qquad \text{in } (0,\ell), \tag{6.5}$$

an identity being valid in the distributional or L^2–sense, since $\vartheta \in W^2$.

In terms of test functions with their compact supports, it is not possible to specify ϑ in further details, one has to take BV into account. Let us reduce the orthogonality relation (6.4) to boundary terms. This has to be done for every case of $\hat{A}$ or $\hat{s}$ separately. Moreover, for convenience we set from now on $\ell := 1$ without restriction of generality. As example we choose case (b)–(c) with the BV for all $\xi \in \mathrm{dom}(\hat{s})$:

$$\xi(0) = 0, \quad \xi'(0) = 0, \quad \xi''(1) = 0, \quad \xi'''(1) = 0.$$

For the above $\vartheta \in \mathrm{dom}(s)$: $\vartheta(0) = 0$ and $\vartheta'(0) = 0$. Inserting (6.5) leads with the BV $\xi(0) = 0$ and $\xi'(0) = 0$ for ξ and doubled partial integration (PI) to

$$\begin{aligned}\langle\xi''|\vartheta''\rangle &= \langle\xi''|a + bx - \frac{\langle 1|\vartheta\rangle}{2}x^2\rangle \\ &\overset{PI}{=} \Big[\overline{\xi'(x)}\big(a + bx - \frac{\langle 1|\vartheta\rangle}{2}x^2\big)\Big]_0^1 - \Big[\overline{\xi(x)}\big(b - \langle 1|\vartheta\rangle x\big)\Big]_0^1 - \langle\xi|1\rangle\langle 1|\vartheta\rangle \\ &= \overline{\xi'(1)}\big(a + b - \frac{\langle 1|\vartheta\rangle}{2}\big) - \overline{\xi(1)}\big(b - \langle 1|\vartheta\rangle\big) - \langle\xi|1\rangle\langle 1|\vartheta\rangle.\end{aligned}$$

Noting $\langle\xi'|1\rangle = \int_0^1 \overline{\xi'(x)}\,dx = \overline{\xi(1)}$ and $\langle 1|\vartheta'\rangle = \int_0^1 \vartheta'(x)\,dx = \vartheta(1)$ by the BV, now the orthogonality relation (6.4) reads as

$$0 = \overline{\xi'(1)}\left(a + b - \frac{\langle 1|\vartheta\rangle}{2}\right) + \overline{\xi(1)}\left(\vartheta(1) - b + \langle 1|\vartheta\rangle\right), \qquad \forall\xi \in \mathrm{dom}(\hat{s}).$$

The expressions in the round brackets vanish because of the following reason: By the boundary extension theorem, e.g. Ref. 16 § 14, 6.6, to all given BV $\kappa^{(m)}(0)$ and $\kappa^{(n)}(1)$ there exists a function $\kappa \in C^\infty([0,1]) \subset W^4$ satisfying the specified BV. That means, when varying ξ in $\mathrm{dom}(\hat{s})$, then $\xi'(1)$ and $\xi(1)$ take arbitrary values independently of each other, and so these expressions have to vanish,

$$2a + 2b - \langle 1|\vartheta\rangle = 0, \qquad -b + \langle 1|\vartheta\rangle + \vartheta(1) = 0. \tag{6.6}$$

On the other hand, (6.5) implies with the BV $\vartheta(0) = 0$ and $\vartheta'(0) = 0$ that

$$\vartheta(x) = \frac{a}{2}x^2 + \frac{b}{6}x^3 - \frac{\langle 1|\vartheta\rangle}{24}x^4. \tag{6.7}$$

As a first consequence we get for $x = 1$,

$$-12a - 4b + \langle 1|\vartheta\rangle + 24\vartheta(1) = 0. \tag{6.8}$$

And when integrating (6.7) and factor out $\int_0^1 \vartheta(x)\,dx = \langle 1|\vartheta\rangle$ one arrives at

$$-20a - 5b + 121\langle 1|\vartheta\rangle = 0. \tag{6.9}$$

Together (6.6), (6.8), and (6.9) build the system of linear equations

$$\begin{pmatrix} 2 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \\ -12 & -4 & 1 & 24 \\ -20 & -5 & 121 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ \langle 1|\vartheta\rangle \\ \vartheta(1)\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

which is uniquely solvable because of a nonzero determinant, and thus

$$0 = a = b = \langle 1|\vartheta\rangle = \vartheta(1) \quad \overset{(6.7)}{\Rightarrow} \quad \vartheta = 0.$$

This means, there does not exist a vector $0 \ne \vartheta \in \mathrm{dom}(s)$, which is orthogonal to $\mathrm{dom}(\hat{s})$ with respect to $\langle .|.\rangle_t$. This is a contradiction to our above assumption that the closure $\bar{\hat{s}}$ of the form $\hat{s}$ of PART (B) does not agree with the closed, positive form s from PART (A). In other words, $\bar{\hat{s}} = s$ with $\mathrm{dom}(\bar{\hat{s}}) = \mathrm{dom}(s)$.

The other support cases work analogously. But for (a)–(b), (b)–(b), and (c)–(c) one may arrive faster at the aim $\vartheta = 0$ with the following argumentation: Remark first that for (a)–(b) and (b)–(b) it is $\langle 1|\vartheta'\rangle = \int_0^1 \vartheta'(x)\,dx = \vartheta(1) - \vartheta(0) = 0$ because of the BV $\vartheta(0) = 0 = \vartheta(1)$ for the orthogonal $\vartheta \in \mathrm{dom}(s)$. Then find all polynomials $p(x)$ up to degree 4, which fulfill the associated 4 BV. Of course $p \in \mathrm{dom}(\hat{s})$, and $\langle p''|\vartheta''\rangle = \langle p''''|\vartheta\rangle$ by double partial integration analogously to (6.1). Inserting p into the orthogonality relation (6.4) yields $0 = \langle p''''|\vartheta\rangle + \langle p'|1\rangle\langle 1|\vartheta'\rangle + \langle p|1\rangle\langle 1|\vartheta\rangle$. Noting that $\langle p''''|\vartheta\rangle = c\langle 1|\vartheta\rangle$ with some constant $c \in \mathbb{C}$, then one arrives at $\langle 1|\vartheta'\rangle = 0 = \langle 1|\vartheta\rangle$, simplifying (6.5) to $\vartheta''(x) = a + bx$, and thus we end up with a simpler system of linear equations.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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